John von Neumann and Hilbert's School of Foundations of Mathematics*

Roman Murawski
Adam Mickiewicz University
Faculty of Mathematics and Comp. Sci.
ul. Umultowska 87
61–614 Poznań, Poland
E-mail: rmur@amu.edu.pl

Abstract

The aim of the paper is to describe main achievements of John von Neumann in the foundations of mathematics and to indicate his connections with Hilbert's School. In particular we shall discuss von Neumann's contributions to the axiomatic set theory, his proof of the consistency of a fragment of the arithmetic of natural numbers and his discovery (independent of Gödel) of the second incompleteness theorem.

1 Introduction

Contacts of John (then still Janos, later Johann) von Neumann with David Hilbert and his school began in the twenties of the 20th century. Being formally a student of the University of Budapest (in fact he appeared there only to pass exams) he was spending his time in Germany and in Switzerland studying there physics and chemistry as well as visiting Hilbert in Göttingen (to discuss with him mathematics). After graduating in chemistry in ETH in Zurich (1925) and receiving the doctorate in Budapest (1926) (his doctoral dissertation was devoted to the axiomatization of set theory — cf. below), he became *Privatdozent* at the University in Berlin (1927–1929), and next in Hamburg (1929–1930). In 1930 he left Germany and went to the USA.[1]

*The support of the Committee for Scientific Research (grant no. 1 H01A 042 27) is acknowledged.

[1] We are not describing further the life of von Neumann and stop at about 1930 because his disappointment with the investigations in the foundations of mathematic let to the fact that after 1930 he lost the interest in the foundational problems and turned his attention to other parts of mathematics, in particular to its applications (see Section 4). Note only that in 1930–1931 von Neumann was visiting lecturer at Princeton

Talking about Hilbert's School we mean the group of mathematicians around Hilbert working in the foundations of mathematics and in the metamathematics (proof theory) in the frameworks of Hilbert's programme of justification of the classical mathematics (by finitistic methods).[2]

Main works of von Neumann from the period from 1922 (the data of his first publication) till 1931 concern mainly metamathematics as well as the quantum mechanics and the theory of operators (also Hilbert worked at that time in just those domains). In this paper we shall be interested in the former, i.e., works devoted to and connected with Hilbert's metamathematical programme.

Recall (to make clearer further considerations) that one of the steps in the realization of Hilbert's programme was the formalization (and in particular the axiomatization) of the classical mathematics (this was necessary for further investigations of mathematical theories by finitistic methods of the proof theory).

Main achievements of von Neumann connected with the ideology of Hilbert's School are the following:

- axiomatization of set theory and (connected with that) elegant theory of the ordinal and cardinal numbers as well as the first strict formulation of principles of definitions by the transfinite induction,
- the proof (by finitistic methods) of the consistency of a fragment of the arithmetic of natural numbers,
- the discovery and the proof of the second incompleteness theorem (this was done independently of Gödel).

The rest of the paper will be devoted just to those items.

2 Foundations of set theory

Contribution of von Neumann devoted to the set theory consisted not only of having proposed a new elegant axiomatic system (extending the system of Zermelo-Fraekel ZFC) but also of having proposed several innovations enriching the system ZFC, in particular the definition of ordinal and cardinal numbers and the theory of definitions by transfinite induction.

The definition of ordinals and cardinals was given by von Neumann in the paper "Zur Einführung der transfiniten Zahlen" (1923) — it was his second publication. He has given there a definition of an ordinal number which could "give unequivocal and concrete form

University in New Jersey, later a professor there. Since 1933 he was professor in the Institute for Advanced Study in Princeton. He died in 1957 at the age of 54.

[2]There is a rich literature on Hilbert's programme — see, e.g., (Murawski, 1999) and the literature indicated there.

to Cantor's notion of ordinal number" in the context of axiomatized set theories (cf. von Neumann, 1923). Von Neumann's ordinal numbers are — using the terminology of G. Cantor — representatives of order types of well ordered sets. In (1923) von Neumann wrote:

> What we really wish to do is to take as the basis of our considerations the proposition: 'Every ordinal is the type of the set of all ordinals that precede it.' But, in order to avoid the vague notion 'type', we express it in the form: 'Every ordinal is the set of the ordinals that precede it.' This is not a proposition proved about ordinals; rather, it would be a definition of them if transfinite induction had already been established.[3]

In this way one obtains the sequence $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \ldots$ — i.e., von Neumann's ordinal numbers.

Those sets — as representatives — are in fact very useful, especially in the axiomatic set theory because they can be easily defined in terms of the relation $\in$ only and they are well order by the relation $\in$. They enable also an elegant definition of cardinal numbers. In the paper (1928) one finds the following definition: a well ordered set M is said to be an ordinal (number) if and only if for all $x \in M$, x is equal to the initial segment of M determined by x itself (as von Neumann wrote: $x = A(x; M)$). Elements of ordinal numbers are also ordinal numbers. An ordinal number is said to be a cardinal number if and only if it is not equipollent to any of its own elements.

In the paper (1923) von Neumann presupposed the notions of a well ordered set and of the similarity and then proved that for any well ordered set there exists a unique ordinal number corresponding to it. All that was done in a naïve set theory but a remark was added that it can be done also in an axiomatic set theory. And in fact von Neumann did it in papers (1928) and (1928a). To be able to do this in a formal way one needs the Axiom of Replacement (in the paper (1923) von Neumann called it Fraenkel's axiom). Since that time von Neumann was an staunch advocate of this axiom.

The problem of definitions by transfinite induction was considered by von Neumann in the paper "Über die Definition durch transfinite Induction, und verwandte Fragen der allgemeinen Mengenlehre" (1928). He showed there that one can always use definitions by induction on ordinal numbers and that such definitions are unequivocal. He proved that for any given condition $\varphi(x, y)$ there exists a unique function f whose domain consists of ordinals such that for any ordinal α one has $f(\alpha) = \varphi(F(f, \alpha), \alpha)$ where $F(f, \alpha)$ is a graph of the function f for arguments being elements of α.

[3]Wir wollen eigentlich den Satz: „Jede Ordnungszahl ist der Typus der Menge aller ihr vorangehenden Ordnungszahlen" zur Grundlage unserer Überlegungen machen. Damit aber der vage Begriff „Typus" vermieden werde, in dieser Form: „Jede Ordnungszahl ist die Menge der ihr vorangehenden Ordnungszahlen." Dies ist kein bewiesener Satz über Ordnungszahlen, es wäre vielmehr, wenn die tranfinite Induktion schon begründet wäre, eine Definition derselben.

Why is the discussed paper so important? For many years, in fact since the axiomatization of set theory by Zermelo, there were no formal counterparts of ordinal and cardinal numbers and this was the reason of avoiding them in the axiomatic set theory. It became a custom to look for ways of avoiding transfinite numbers and transfinite induction in mathematical reasonings (cf., e.g., Kuratowski, 1921 and 1922). Von Neumann's paper introduced a new paradigm which works till today. The leading idea of the paper was the will to give set theory as wide field as possible.

The most important and known contribution of John von Neumann is undoubtedly a new approach and new axiomatization of set theory. The main ideas connected with that appeared by von Neumann already in 1923 (he was then 23!!!). He described them in a letter to Ernst Zermelo from August 1923.[4] He wrote there that the impulse to his ideas came from a work by Zermelo "Untersuchungen über die Grundlagen der Mengenlehre. I" (1908) and added that in some points he went away from Zermelo's ideas, in particular

- the notion of 'definite property' had been avoided — instead the "acceptable schemas" for the construction of functions and sets had been presented,
- the axiom of replacement had been assumed — it was necessary for the theory of ordinal numbers (later von Neumann emphasized, like Fraenkel and Skolem, that it is needed in order to establish the whole series of cardinalities — cf. von Neumann 1928a),
- sets that are "too big" (for example the set of all sets) had been admitted but they were taken to be inadmissible as elements of sets (that sufficed to avoid the paradoxes).

About 1922–1923 while preparing a paper in which those ideas should be developed he contacted Abraham Fraenkel. The latter recalled this (already after the death of von Neumann) in a letter to Stanisław Ulam in such a way:[5]

> Around 1922–23, being then professor at Marburg University, I received from Professor Erhard Schmidt, Berlin (on behalf of the *Redaktion* of the *Mathematische Zeitschrift*) a long manuscript of an author unknown to me, Johann von Neumann, with the title "Die Axiomatisierung der Mengenlehre", this being his eventual doctor[al] dissertation which appeared in the *Zeitschrift* only in 1928 (vol. 27). I was asked to express my views since it seemed incomprehensible. I don't maintain that I understood everything, but enough to see that this was an outstanding work and to recognize *ex ungue leonem*. While answering in this sense, I invited the young scholar to visit me (in Marburg) and discussed things with him, strongly advising him to prepare the ground for the understanding of

[4] This letter was partly reproduced in Meschowski, 1967, 289–291.

[5] Letter from Fraenkel to Ulam in (Ulam, 1958).

so technical an essay by a more informal essay which should stress the new access to the problem and its fundament consequences. He wrote such an essay under the title "Eine Axiomatisierung der Mengenlehre" and I published it in 1925 in the *Journal für Mathematik* (vol. 154) of which I was then Associate Editor.

Before continuing the story let us explain that *ex ungue leonem* — spotting a lion from the claw — is an expression used by Daniel Bernoulli while talking about Newton two and a half centuries ago. Bernoulli was namely sent a mathematical paper without a name of the author but he immediately recognized that it has been written just by Newton.

Von Neuman began the paper "Eine Axiomatisierung der Mengenlehre" (1925) by writing:

> The aim of the present work is to give a logically unobjectionable axiomatic treatment of set theory. I would like to say something first about difficulties which make such an axiomatization of set theory desirable.[6]

He stressed explicitly three points mentioned in the letter to Zermelo.

The characteristic feature of the system of set theory proposed by von Neumann is the distinction between classes, "domains" (*Bereiche*) and sets (*Mengen*). Classes are introduced by the Principle of Comprehension — von Neumann seems to have regarded this principle as the quintessence of what he called "naïve set theory" (cf. von Neumann 1923, 1928, 1929). His appproach to set theory was strongly based on the idea of limitation of size according to which: a class is a set if and only if it is not "too big". The latter notion was described by the following axiom:

($*$) *A class is "too big"* (in the terminology of (Gödel, 1940) — *is a proper class*) *if and only if it is equivalent to the class of all things.*

Hence a class of the cardinality smaller than the cardinality of the class of all sets is a set. Von Neumann states further that the above principle implies both the Axiom of Separation and the Axiom of Replacement. It implies also the well ordering theorem (he indicated it already in the letter to Zermelo). Indeed, according to the reasoning used in the Burali-Forti paradox, the class On of all ordinal numbers is not a set, hence by the above principle it is equipollent with the class V of all sets. In this way one obtains a strengthened version of the well ordering theorem, namely:

The class V of all sets can be well ordered.

In the paper "Die Axiomatisierung der Mengenlehre" (1928a) [this was in fact a "mathematical" version of the system of set theory announced in the paper (1925)] von Neumann

[6] Das Ziel der vorliegenden Arbeit ist, eine logisch einwandfreie axiomtische Darstellung der Mengenlehre zu geben. Ich möchte dabei einleitend einiges über die Schwierigkeiten sagen, die einen derartigen Aufbau der Mengenlehre erwünscht gemacht haben.

observed that the principle gives also a global choice function F such that for any nonempty set A it holds: $F(A) \in A$.

The Axiom of Choice, being a consequence of $(*)$, enabled von Neumann to introduce ordinal and cardinal numbers without the necessity of introducing any new primitive notions. It was in fact a realization of the idea he wrote about in 1923. He used here the Fraenkel's Axiom of Replacement.

Observe that the distinction between classes and sets appeared already by Georg Cantor — he wanted to eliminate in this way the paradox of the set of all sets. Cantor used to call classes "absolutely infinite multiplicities". But he gave no precise criterion of distinguishing classes and sets — it was given only by von Neumann. The latter has also shown that Cantor was mistaken when he claimed that the absolutely infinite multiplicities (e.g., the multiplicity of all ordinal numbers) cannot be treated as consistent objects.

In the original formulation of set theory by von Neumann there are no notions of a set and a class. Instead one has there the primitive notion of a function (and of the relation $\in$). Von Neumann claimed that it is in fact only a technical matter — indeed, the notions of a set and of a function are mutually definable, i.e., a set can be treated as a function with values 0 and 1 (the characteristic function of the set) and, vice versa, a function can be defined as a set of ordered pairs.

Add that in the von Neumann's system of set theory there are no urelements — there are only pure sets and classes. On the other hand among axioms there is the Axiom of Foundation introduced by Dimitri Mirimanoff in (1917).[7] This axiom garantees that there are no infinite decreasing $\in$-sequences, i.e., such sequences that $\ldots \in x_n \in \ldots \in x_1 \in x_0$ and that there are neither sets x such that $x \in x$ nor sets x and y such that $x \in y$ and $y \in x$. This axiom implies that the system of set theory containing it becomes similar to the theory of types: one can say that the system ZF with the Axiom of Foundation can be treated as an extension of the (cumulative) theory of types to the transfinite types described in a simpler language than it was the case by Russell.

It should be noticed that von Neumann was one of the first who investigated metatheoretical properties of the axiomatic set theory. In particular he studied his own system from the point of view of the categoricity (1925) and of the relative consistency (1929). Probably he was also the first author who called attention to the Skolem paradox. According to von Neumann this paradox stamps axiomatic set theory "with the mark of unreality" and gives reasons to "entertain reservations" about it (cf. 1925).

Von Neumann wrote about the proof of the relative consistency of his system of set theory in the paper "Über eine Widerspruchsfreiheitsfrage der axiomatischen Mengenlehre" (1929). He saw main difficulties in the axiom $(*)$. Therefore he considered two axiomatic systems:

[7] In fact it was for the first time discussed by Mirimanoff and Skolem it was just von Neumann who as the first formulated it explicitely.

system S which was his original system (hence with the axiom $(*)$) and the system S^* which was von Neumann's system but with the Axiom of Replacement and the Axiom of Choice instead of the axiom $(*)$. In the paper (1929) he proved that:

1. S^* will remain consistent if one adds the Axiom of Foundation and does not admit urlements,
2. S^* is a subsystem of such a system.

Hence von Neumenn proved the relative consistency of the Axiom of Foundation with respect to the system S^*. It was in fact the first significant metatheoretical result on set theory.

It is worth saying that in (1929) von Neumann developed the cumulative hierarchy in technical details. Using the Axiom of Foundation and the ordinal numbers he showed that the universe of sets can be divided into "levels" indexed by ordinal numbers. He introduced the notion of a rank of a set: a rank of a set x is the smallest ordinal number α such that the set x appears at the level α. This hierarchy is cumulative, i.e., lower levels are included in higher ones. The hierarchy can be precisely defined as follows:

$$\begin{aligned} V_0 &= \emptyset, \\ V_{\alpha+1} &= V_\alpha \cup \mathcal{P}(V_\alpha), \\ V_\lambda &= \bigcup_{\alpha<\lambda} V_\alpha \quad \text{for } \lambda \in \lim, \\ V &= \bigcup_{\alpha\in\mathrm{On}} V_\alpha, \\ \mathrm{rank}(x) &= \mu\alpha(x \in V_\alpha). \end{aligned}$$

It is worth adding here that von Neumann treated the Axiom of Foundation rather as a tool in the metatheoretical investigations of set theory.

We are talking the whole time about axioms of set theory but no axioms have been given so far. It is time to do it!

Let us start by stating that the main idea underlying von Neumann's system of set theory has been accepted with enthusiasm — in fact it provided a remedium to too drastic restrictions put on objects of set theory by the system ZF of Zermelo-Fraenkel (one was convinced that such strong restrictions are not needed in order to eliminate paradoxes; on the other hand the restrictions put by ZF made the development of mathematics within ZF very difficult and unnatural). Nevertheless the system of von Neumann was not very popular among specialists — the reason was the fact that it was rather counter-intuitive and was based on a rather difficult notions (recall that the primitive notion of a function instead of the notion of a set was used there). Hence the need of reformulating the original system. That has been done by Paul Bernays: in (1937) he announced the foundations, and in a series of papers published in the period 1937–1958 (cf. 1937, 1941, 1942, 1958) he gave an extensive axiomatic system of set theory which realized the ideas of von Neumann and simultaneously he suceeded to formulate his system in a language close to the language of the system ZF.

In (1937) he wrote:

> The purpose of modifying the von Neumann system is to remain nearer to the structure of the original Zermelo system and to utilize at the same time some of the set-theoretic concepts of the Schröder logic and of *Principia Mathematica* which have become familiar to logicians. As will be seen, a considerable simplification results from this arrangement.

The universe of set theory consists by Bernays of two parts:

- sets denoted by $x, y, z, \ldots$,
- classes denoted by $A, B, C, \ldots$

Hence it is not an elementary system! There are two primitive notions: $\in$ (= to be an element of (to belong to) a set) and η (= to be an element (to belong to) a class). Hence one has two types of atomic formulas: $x \in y$ and $x \eta A$. There are also two groups of axioms: axioms about sets (they are analogous to axioms of Zermelo) and axioms characterizing classes. The very important feature of Bernays' axioms is the fact that there are only finitely many axioms and there are no axiom schemes.

In a work devoted to the consistency of the Axiom of Choice and of the Generalized Continuum Hypothesis K. Gödel gave an axiomatic system of set theory which is in fact a modification of Bernays' system. Its main advantage is that it is an elementary system (i.e., it contains only one type of variables).[8]

Let us describe now in details the system NBG of Gödel. It is based on the idea that the variables vary over classes. Among classes we distinguish those classes that are elements of other classes. They are called sets and their totality is denoted by V. The remaining classes are called proper classes.

Define the class V as follows

$$x \in V \longleftrightarrow (\exists y)(x \in y).$$

Hence x is a set if and only if there exists a class y such that $x \in y$. Define also a notion of a function in the following way:

$$\mathrm{Func}(r) \longleftrightarrow \forall x \forall y \forall z [(x, y) \in r \wedge (x, z) \in r \longrightarrow y = z].$$

The system NBG is based on the following nonlogical axioms:

[8] It is worth noting here that the idea of using in the system of von Neumann – Bernays only one type of variables and one membership relation is due to Alfred Tarski — cf. (Mostowski, 1939, p. 208) and (Mostowski, 1949, p. 144).

- (Extensionality)
$$\forall x \,\forall y \,[\forall z(z \in x \longleftrightarrow z \in y) \longrightarrow x = y],$$
- (Axiom of Classes)
$$\exists x \,\forall y \,[\exists z(y \in z) \longrightarrow y \in x],$$
- (Axiom of the Empty Set)
$$\exists x \,[\forall y \,(y \notin x) \wedge \exists z \,(x \in z)],$$
- (Pairing Axiom)
$$\forall x \in V \,\forall y \in V \,\exists z \in V \,[\forall u(u \in z \longleftrightarrow u = x \vee u = y)],$$
- (Axiom Scheme of Class Existence) if Φ is a formula with free variables $v_1, \ldots, v_n$, then the following formula
$$\forall v_1, \ldots, v_n \in V \,\exists z \,\forall x \,[x \in z \longleftrightarrow (x \in V \wedge \Phi^{(V)}(x, v_1, v_2, \ldots, v_n))]$$
is an axiom (note that one cannot quantifier in Φ over class variables!); $\Phi^{(V)}$ denotes the relativization of Φ to the class V,
- (Axiom of Union)
$$\forall x \in V \,\exists y \in V \,\forall u \,[u \in y \longleftrightarrow \exists v(u \in v \wedge v \in x)],$$
- (Power Set Axiom)
$$\forall x \in V \,\exists y \in V \,\forall u \,(u \in y \longleftrightarrow u \subseteq x),$$
- (Infinity Axiom)
$$\exists x \in V \,[\emptyset \in x \wedge \forall u \in x \,\forall v \in x \,(u \cup \{v\} \in x)],$$
- (Axiom of Replacement)
$$\forall x \in V \,\forall r \,[Func(r) \longrightarrow \exists y \in V \,\forall u \,(u \in y \longleftrightarrow \exists v \in x \,((v, u) \in r)],$$
- (Axiom of Foundation)
$$\forall x[x \neq \emptyset \longrightarrow \exists y \in x \,(x \cap y = \emptyset)].$$

We one adds to this system the following Axiom of Global Choice (in a strong version):

$$\exists x[Func(x) \wedge \forall y \in V[y \neq \emptyset \longrightarrow \exists z(z \in y \wedge (y, z) \in x)]]$$

then one obtains the system denoted as NBGC.

It has turned out that the axiom scheme of class existence can be replaced by the following (finitely many!) axioms:

$$\exists a\, \forall x, y \in V[(x, y) \in a \longleftrightarrow x \in y]$$

(it says that a jest a graph of the membership relation $\in$ for sets),

$$\forall a\, \forall b\, \exists c\, \forall x[x \in c \longleftrightarrow (x \in a \wedge x \in b)]$$

(it defines the intersection of classes),

$$\forall a\, \exists b\, \forall x \in V\, [x \in b \longleftrightarrow x \notin a]$$

(it defines the complement of a class),

$$\forall a\, \exists b\, \forall x \in V\, [x \in b \longleftrightarrow \exists y \in V((x, y) \in a)]$$

(it defines the left domain of a relation),

$$\forall a\, \exists b\, \forall x, y \in V\, [(x, y) \in b \longleftrightarrow x \in a),$$

$$\forall a\, \exists b\, \forall x, y, z \in V\, [(x, y, z) \in b \longleftrightarrow (y, z, x) \in a],$$

$$\forall a\, \exists b\, \forall x, y, z \in V\, [(x, y, z) \in b \longleftrightarrow (x, z, y) \in a].$$

The systems NBG and NBGC have very nice metamathematical properties, in particular:

- NBG (NBGC) is finitely axiomatizable (observe that the Zermelo-Fraenkel system ZF is not finitely axiomatizable!),

- NBG is a conservative extension of ZF with respect to formulas saying about sets, i.e., for any formula φ of the language of set theory:

  $$\mathrm{ZF} \vdash \varphi \qquad \text{if and only if} \qquad \mathrm{NBG} \vdash \varphi^{(V)}$$

  where $\varphi^{(V)}$ denotes the relativization of φ to the class V of all sets (and similarly for NBGC and ZFC where the latter symbol denotes the theory ZF plus the Axiom of Choice AC),

- NBG is consistent if and only if ZF is consistent (and similarly for NBGC and ZFC).

3 Consistency proof for arithmetic

One of the main aims of Hilbert's programme was the consistency proof (by save finitary methods) for the whole classical mathematics. Students of Hilbert took this task and soon first partial results appeared. The first work in this direction was the paper by Wilhelm Ackermann (1924) where he gave a finitistic proof of the consistency of arithmetic of natural numbers without the axiom (scheme) of induction.[9]

Next attempt to solve the problem of the consistency was the paper "Zur Hilbertschen Beweistheorie" (1927) by von Neumann. He used another formalism than that in (Ackermann, 1924) and, similarly as Ackermann, proved in fact the consistency of a fragment of arithmetic of natural numbers obtained by putting some restrictions on the induction. We cannot consider here the (complicated) technical details of von Neumann's proof. It is worth mentioning that in the introductory section of von Neumann's (1927) a nice and precise formulation of aims and methods of Hilbert's proof theory was given. It indicated how was at that time the state of affairs and how Hilbert's programme was understood. Therefore we shall quote the appropriate passages.

Von Neumann writes that the essential tasks of proof theory are (cf. von Neumann, 1927, 256–257):

> I. First of all one wants to give a proof of the consistency of the classical mathematics. Under 'classical mathematics' one means the mathematics in the sense in which it was understood before the begin of the criticism of set theory. All settheoretic methods essentially belong to it but not the proper abstract set theory. [...]
>
> II. To this end the whole language and proving machinary of the classical mathematics should be formalized in an absolutely strong way. The formalism cannot be too narrow.
>
> III. Then one must prove the consistency of this system, i.e., one should show that certain formulas of the formalism just described can never be "proved".
>
> IV. One should always strongly distinguish here between various types of "proving": between formal ("mathematical") proving in a given formal system and contents ("metamathematical") proving [of statements] about the system. Whereas the former one is an arbitrarily defined logical game (which should to a large extent be analogues to the classical mathematics), the latter is a chain of directly evident contents insights. Hence this "contents proving" must proceed according to the intuitionistic logic of Brouwer and

[9] In fact it was a much weaker system than the usual system of arithmetic but the paper provided the first attempt to solve the problem of consistency. Later in the paper (1940) Ackermann proved the consistency of the full arithmetic of natural numbers by using methods from his paper (1924) and the transfinite induction.

> Weyl. Proof theory should so to speak construct classical mathematics on the intuitionistic base and in this way lead the strict intuitionism ad absurdum.[10]

Note that von Neumann identifies here finitistic methods with intuitionistic ones. This was then current among members of the Hilbert's school. The distinction between those two notions was to be made explicit a few years later — cf. (Hilbert and Bernays, 1934, pp. 34 and 43) and (Bernays 1934, 1935, 1941a), see also (Murawski, 2001).

As an interesting detail let us add that on the paper (1927) by von Neumann reacted critically Stanisław Leśniewski publishing the paper "Grundzüge eines neuen Systems der Grundlagen der Mathematik" (1929) in which he critically analyzed various attempts to formalize logic and mathematics. Leśniewski among others expresses there his doubts concerning the meaning and significance of von Neumann's proof of the consistency of (a fragment of) arithmetic and constructs — to maintain his thesis — a "counterexample", namely he deduce (on the basis of von Neumann's system) two formulas a and $\neg a$, hence an inconsistency.

Von Neumann answered to Leśniewski's objections in the paper "Bemerkungen zu den Ausführungen von Herrn St. Leśniewski über meine Arbeit 'Zur Hilbertschen Beweistheorie'"(1931). Analyzing the objections of Leśniewski he came to the conclusion that there is in fact a misunderstanding resulting from various ways in which they both understand principles of formalization. He used also the occasion to fulfil the gap in his paper (1927).

Add also that looking for a proof of the consistency of the classical mathematics and being (still) convinced of the possibility of finding such a proof (in particular a proof of the consistency of the theory of real numbers) von Neumann doubted whether there are any chances to find such a proof for the set theory — cf. his paper (1929).

[10] I. In erster Linie wird der Nachweis der Widerspruchsfreiheit der klassischen Mathematik angestrebt. Unter „klassischer Mathematik" wird dabei die Mathematik in demjenigen Sinne verstanden, wie sie bis zum Auftreten der Kritiker der Mengenlehre anerkannt war. Alle mengentheoretischen Methoden gehören im wesentlichen zu ihr, nicht aber die eigentliche abstrakte Mengenlehre. [...]

II. Zu diesem Zwecke muß der ganze Aussagen- und Beweisapparat der klassischen Mathematik absolut streng formalisiert werden. Der Formalismus darf keinesfalls zu eng sein.

III. Sodann muß die Widerspruchsfreiheit dieses Systems nachgewiesen werden, d.h. es muß gezeigt werden, daß gewisse Aussagen „Formeln" innerhalb des beschriebenen Formalismus niemals „bewiesen" werden können.

IV. Hierbei muß stets scharf zwischen verschiedenen Arten des „Beweisens" unterschieden werden: Dem formalistischen („mathematischen") Beweisen innerhalb des formalen Systems, und dem inhaltlichen („metamathematischen") Beweisen über das System. Während das erstere ein willkürlich definiertes logisches Spiel ist (das freilich mit der klassischen Mathematik weitgehend analog sein muß), ist das letztere eine Verkettung unmittelbar evidenter inhaltlicher Einsichten. Dieses „inhaltliche Beweisen" muß also ganz im Sinne der Brouwer-Weylschen intuitionistischen Logik verlaufen: Die Beweistheorie soll sozusagen auf intuitionistischer Basis die klassische Mathematik aufbauen und den strikten Intuitionismus so ad absurdum führen.

4 Von Neumann and Gödel's second incompleteness theorem

How much von Neumann was engaged in the realization of Hilbert's programme and how high was his position in this group can be judged from the fact that just he has been invited by the organizers of the Second Conference on the Epistemology of Exact Sciences (organized by Die Gesellschaft für Empirische Philosophie)[11] held in Königsberg, 5–7th September 1930, to give a lecture presenting formalism — one of the three main trends in the contemporary philosophy of mathematics and the foundations of mathematics founded by Hilbert. The other two main trends: logicism and intuitionism were presented by Rudolf Carnap and Arend Heyting, resp.

In his lecture "Die formalistische Grundlegung der Mathematik"(cf. 1931a) von Neumann recalled basic presuppositions of Hilbert's programme and claimed that thanks to the works of Russell and his school a significant part of the tasks put by Hilbert has already been realized. In fact the unique task that should be fulfilled now is "to find a finitistically combinatorial proof of the consistency of the classical mathematics". And he added that this task turned out to be difficult. On the other hand, partial results obtained so far by W. Ackermann, H. Weyl and himself make possible to cherish hopes that it can be realized. He finished his lecture by saying: "Whether this can be done for a more difficult and more important system of [the whole] classical mathematics will show the future."

On the last day of the conference, i.e., on 7th September 1930, a young Austrian mathematician Kurt Gödel announced his recent (not yet published) results on the incompleteness of the system of arithmetic of natural numbers and richer systems.

It seems that the only participant of the conference in Königsberg who immediately grasped the meaning of Gödel's theorem and understood it was von Neumann. After Gödel's talk he had a long discussion with him and asked him about details of the proof. Soon after coming back from the conference to Berlin he wrote a letter to Gödel (on 20th November 1930) in which he announced that he had received a remarkable corollary from Gödel's First Theorem, namely a theorem on the unprovability of the consistency of arithmetic in arithmetic itself. In the meantime Gödel developed his Second Incompleteness Theorem and included it in his paper "Über formal unentscheidbare Sätze der 'Principia Mathematica' und verwandter Systeme. I" (cf. Gödel, 1931). In this situation von Neumann decided to leave the priority of the discovery to Gödel.

[11] This conference was organized together with the 91st Convention of the Society of German Scientists and Physicians (Gesellschaft deutscher Naturforscher und Ärzte) and the 6th Conference of German Mathematicians and Physicists (Deutsche Physiker- und Mathematikertagung).

5 Concluding remarks

Gödel's incompleteness results had great influence on von Neumann's views towards the perspectives of investigations on the foundations of mathematics. He claimed that "Gödel's result has shown the unrealizability of Hilbert's program" and that "there is no more reason to reject intuitionism" (cf. his leter to Carnap of 6th June 1931 — see Mancosu, 1999, 39–41). He added in this letter:

> Therefore I consider the state of the foundational discussion in Königsberg to be outdated, for Gödel's fundamental discoveries have brought the question to a completely different level. (I know that Gödel is much more careful in the evoluation of his results, but in my opinion on this point he does not see the connections correctly).

Incompleteness results of Gödel changed the opinions cherished by von Neumann and convinced him that the programme of Hilbert cannot be realized. In the paper "The Mathematician"(1947) he wrote:

> My personal opinion, which is shared by many others, is, that Gödel has shown that Hilbert's program is essentially hopeless.

Another reason for the disappointment of von Neumann's with the investigations in the foundations of mathematics could be the fact that he became aware of the lack of categoricity of set theory, i.e., that there exist various nonisomorphic models of set theory. The latter fact implies that it is impossible to describe the world of mathematics in a unique way. In fact there is no absolute description, all descriptions are relative.

Not only von Neumman was aware of this feature of set theory. Also Fraenkel and Thoralf Skolem realized this. And they have proposed various measures. In particular Fraenkel in his very first article "Über die Zermelosche Begründung der Mengenlehre" (1921) sought to render set theory categorical by introducing his Axiom of Restriction, inverse to the completeness axiom that Hilbert had proposed for geometry in 1899. Whereas Hilbert had postulated the existence of a maximal model satisfying his other axioms, Fraenkel's Axiom of Restriction asserted that the only sets to exist were those whose existence was implied by Zermelo's axioms and by the Axiom of Replacement. In particular, there were no urelements. One should add that Fraenkel did not distinguish properly a language and a metalanguage and confused them.

The approach of Skolem was different — but we will not go into technical details here.[12]

Von Neumann also examined the possible categoricity of set theory. In order to render it as likely as possible that his own system was categorical, he went beyond Mirimanoff

[12] See, e.g., Moore, 1982, Section 4.9.

and augmented it by the axiom stating that there are no infinite descending ∈-sequences. He recognized that his system would surely lack categoricity unless he excluded weakly inaccessible cardinals (i.e., regular cardinals with an index being a limit ordinal). Von Neumann rejected also the Fraenkel's Axiom of Restriction as untenable because it relied on the concept of subdomain and hence on inconsistent "naïve" set theory. He was also aware of the difficulties implied by Löwenheim-Skolem theorem.

Von Neumann treated the lack of categoricity of set theory, certain relativism of it as an argument in favor of intuitionism (cf. his 1925). He stressed also the distance between the naïve and the formalized set theory and called attention to the arbitrariness of restrictions introduced in axiomatic set theory (cf. 1925, 1928a, 1929). He saw also no rescue and no hope in Hilbert's programme and his proof theory — in fact the latter was concerned with consistency and not with categoricity.

One should notice here that von Neumann's analyses lacked a clear understanding of the difference and divergence between first-order and second-order logic and their effects on categoricity. Today it is known, e.g., that Hilbert's axioms for Euclidian geometry and for the real numbers as well as Dedekind-Peano axioms for the arithmetic of natural numbers are categorical in second-order logic and non-categoric in the first-order logic. Only Zermelo (perhaps under the influence of Hilbert[13]) claimed that the first-order logic is insufficent for mathematics, and in particular for set theory. It became the dominant element in Zermelo's publications from the period 1929–1935. It is worth noting here that he spoke about this for the first time in his lectures held in Warsaw in May and June 1929.

After 1931 von Neumann ceased publishing on the mathematical logic and the foundations of mathematics — he came to the conclusion that a mathematician should devote his attention to problems connected with the applications. In (1947) he wrote:

> As a mathematical discipline travels far from its empirical source, or still more, if it is a second and third generation only indirectly inspired by ideas coming from "reality", it is beset with very grave dangers. It becomes more and more purely aestheticizing, more and more purely *l'art pour l'art.* [...] In other words, at a great distance from its empirical source, or after much "abstract" inbreeding, a mathematical subject is in danger of degeneration.

[13] Hilbert and Ackermann wrote in (1928): "As soon as the object of investigation becomes the foundation of ... mathematical theories, as soon as we went to determine in what relation the theory stands to logic and to what extent it can be obtained from purely logical operations and concepts, then second-order logic is essential." In particular they defined the set-theoretic concept of well-ordering by means of second-order, rather than first-order, logic.

6 References

Ackermann W., 1924, Begründung des „tertium non datur" mittels der Hilbertschen Theorie der Widerspruchsfreiheit, *Mathematische Annalen* 93, 1–36.

Ackermann, W., 1940, Zur Widerspruchsfreiheit der Zahlentheorie, *Mathematische Annalen* 117, 162–194.

Bernays, P., 1934, 'Quelques points essentiels de la métamathématiques', *L'Enseignement mathématique* 34, 70–95.

Bernays, P., 1935, Hilberts Untersuchungen über die Grundlagen der Arithmetik, in: Hilbert, D. *Gesammelte Abhandlungen*, Verlag von Julius Springer, Berlin, Bd. 3, pp. 196–216.

Bernays P., 1937, A system of axiomatic set theory. I, *Journal of Symbolic Logic* 2, 65–77.

Bernays P., 1941, A system of axiomatic set theory. II, *Journal of Symbolic Logic* 6, 1–17.

Bernays, P., 1941a, Sur les questions méthodologiques actuelles de la théorie hilbertienne de la démonstration, in: Gonseth, F. (Ed.) *Les entretiens de Zurich sur les fondements et la méthode des sciences mathématiques, 6–9 décembre 1938*, Leemann, Zurich 1941, pp. 144–152; Discussion, pp. 153–161.

Bernays P., 1942, A system of axiomatic set theory. III, *Journal of Symbolic Logic* 7, 65–89, 133–145.

Bernays P., 1958, *Axiomatic Set Theory*, North-Holland Publishing Company, Amsterdam.

Fraenkel A., 1921, Über die Zermelosche Begründung der Mengenlehre, *Jahresbericht der Deutschen Mathematiker-Vereinigung* (Angelegenheiten) 30, 97–98.

Gödel, K., 1931, 'Über formal unentscheidbare Sätze der 'Principia Mathematica' und verwandter Systeme. I', *Monatshefte für Mathematik und Physik* 38, 173–198. Reprinted with English translation 'On Formally Undecidable Propositions of Principia Mathematica and Related Systems' in: Gödel, K. *Collected Works*, vol. I, ed. by Feferman, S. *et al.*, Oxford University Press, New York and Clarendon Press, Oxford, 1986, pp. 144–195.

Gödel K., 1940, *The Consistency of the Axiom of Choice and of the Generalized Continuum Hypothesis with the Axioms of Set Theory*, Princeton University Press, Princeton, N.J.

Hilbert, D. and Ackermann, W., 1928, *Grundzüge der theoretischen Logik*, Springer Verlag, Berlin. English translation: *Principles of Mathematical Logic*, AMS Chelsea Publishing, American Mathematical Society, Providence, Rhode Island 1958.

Hilbert, D. and Bernays, P., 1934–1939, *Grundlagen der Mathematik*, Bd. I 1934, Bd. II 1939, Springer-Verlag, Berlin.

Kuratowski K., 1922, Sur la notion de l'ordre dans la théorie des ensembles, *Fundamenta Mathematicae* 2, 161–171.

Kuratowski K., 1923, Une méthode d'elimination des nombres transfinis des raisonnements mathématiques, *Fundamenta Mathematicae* 3, 76–108.

Leśniewski S., 1929, Grundzüge eines neuen Systems der Grundlagen der Mathematik, *Fundamenta Mathematicae* 14, 1–81.

Mancosu, P., 1999, Between Vienna and Berlin: The Immediate Reception of Gödel's Incompleteness Theorems, *History and Philosophy of Logic* 20, 33–45.

Meschkowski H., 1967, *Probleme des Unendlichen. Werk und Leben Georg Cantors*, Vieweg, Braunschweig. Reproduced with additions as *Georg Cantor. Leben, Werk und Wirkung*, Bibliographisches Institut, Mannheim 1983.

Mirimanoff D., 1917, Les antinomies de Russell et de Burali-Forti et le probléme fondamental de la théorie des ensembles, *L'Enseignement Mathématique* 19, 37–52.

Moore G. H., 1982, *Zermelo's Axiom of Choice. Its Origins, Development, and Influence*, Springer Verlag, New York/Heidelberg/Berlin.

Mostowski A., 1939, Über die Unabhängighkeit des Wohlordnungssatzes vom Ordnungsprinzip, *Fundamenta Mathematicae* 32, 201–252.

Mostowski A., 1949, An undecidable arithmetical statement, *Fundamenta Mathematicae* 36, 143–164.

Murawski R., 1999, *Recursive Functions and Metamathematics*, Kluwer Academic Publishers, Dordrecht/Boston/London 1999.

Murawski R., 2001, On proof of the consistency of arithmetic, *Studies in Logic, Grammar and Rhetoric* 4 (17), 41–50.

von Neumann J., 1923, Zur Einführung der transfiniten Zahlen, *Acta Szeged* 1, 199–208. Reprinted in: von Neumann J., 1961, 24–33.

von Neumann J., 1925, Eine Axiomatisierung der Mengenlehre, *Journal für Mathematik* 154, 219–240. Reprinted in: von Neumann J., 1961, 34–57.

von Neumann J., 1927, Zur Hilbertschen Beweistheorie, *Mathematische Zeitschrift* 26, 1–46. Reprinted in: von Neumann J., 1961, 256–302.

von Neumann J., 1928, Über die Definitionen durch transfinite Induktion, und verwandte Fragen der allgemeinen Mengenlehre, *Mathematische Annalen* 99, 373–391. Reprinted in: von Neumann J., 1961, 320–338.

von Neumann J., 1928a, Die Axiomatisierung der Mengenlehre, *Mathematische Zeitschrift* 27, 669–752. Reprinted in: von Neumann J., 1961, 339–422.

von Neumann J., 1929, Über eine Widerspruchsfreiheitsfrage der axiomatischen Mengenlehre, *Journal für Mathematik* 160, 227–241. Reprinted in: von Neumann J., 1961, 494–508.

von Neumann J., 1931, Bemerkungen zu den Ausführungen von Herrn St. Leśniewski über meine Arbeit "Zur Hilbertschen Beweistheorie", *Fundamenta Mathematicae* 17, 331–334. Reprinted in: von Neumann J., 1961a, 230–233.

von Neumann J., 1931a, Die formalistische Grundlegung der Mathematik, *Erkenntniss* 2, 116–121. Reprinted in: von Neumann J., 1961a, 234–239.

von Neumann J., 1947, The Mathematician, in: *The Works of the Mind*, ed. by R.B. Heywood, University of Chicago Press, Chicago, 180–196. Reprinted in: von Neumann J., 1961, 1–9.

von Neumann J., 1961, *Collected Works*, vol. I: *Logic, Theory of Sets and Quantum Mechanics*, Pergamon Press, Oxford/London/New York/Paris.

von Neumann J., 1961a, *Collected Works*, vol. II: *Operators, Ergodic Theory and Almost Periodic Functions in a Group*, Pergamon Press, Oxford/London/New York/Paris.

Ulam S., 1958, John von Neumann, 1903–1957, *Bulletin of the American Mathematical Society* 64 (May Supplement), 1–49.

Zermelo, E., 1908, Untersuchungen über die Grundlagen der Mengenlehre. I, *Mathematische Annalen* 65, 261–281.